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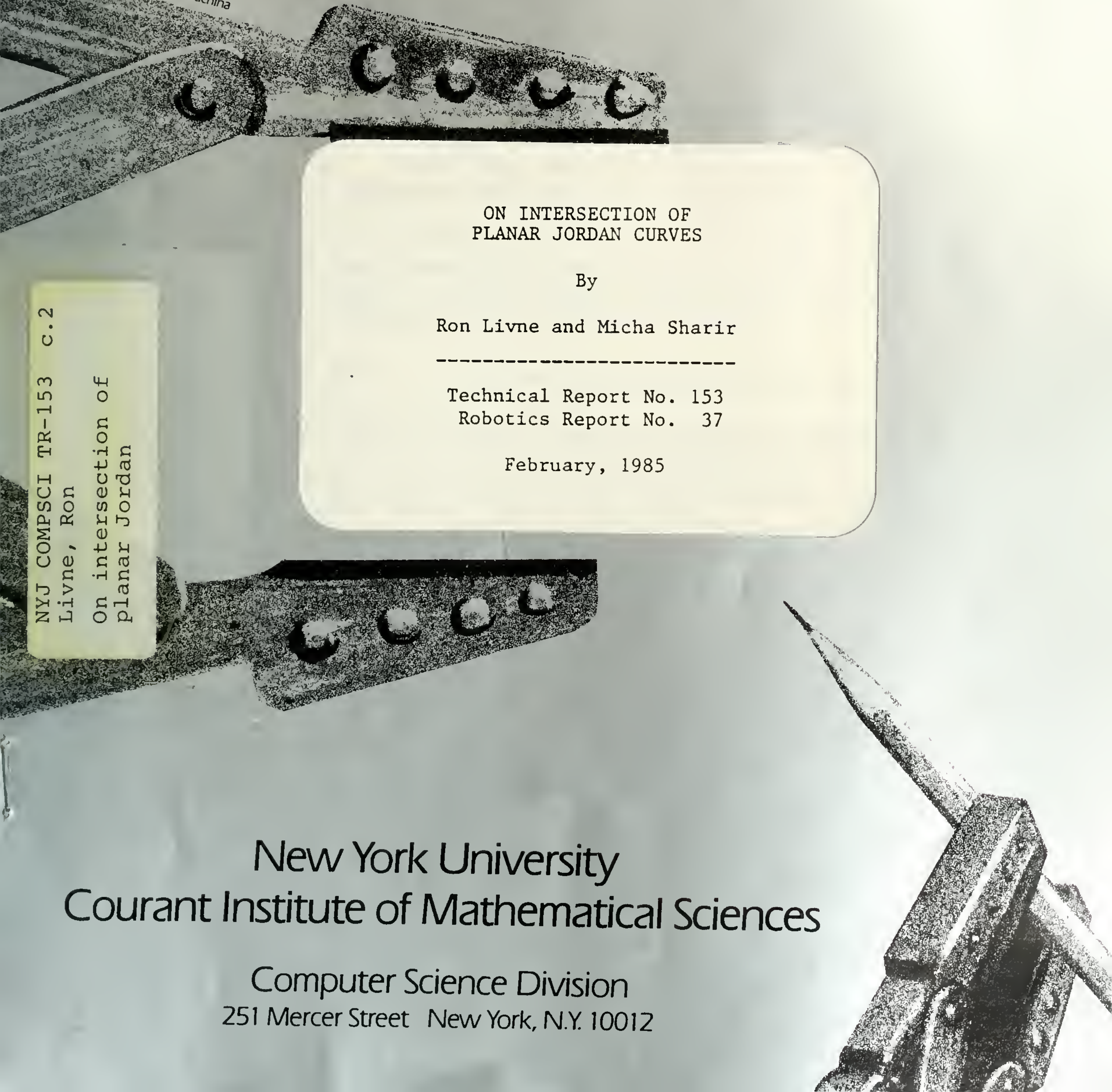

ON INTERSECTION OF PLANAR JORDAN CURVES

By

Ron Livne and Micha Sharir

Technical Report No. 153
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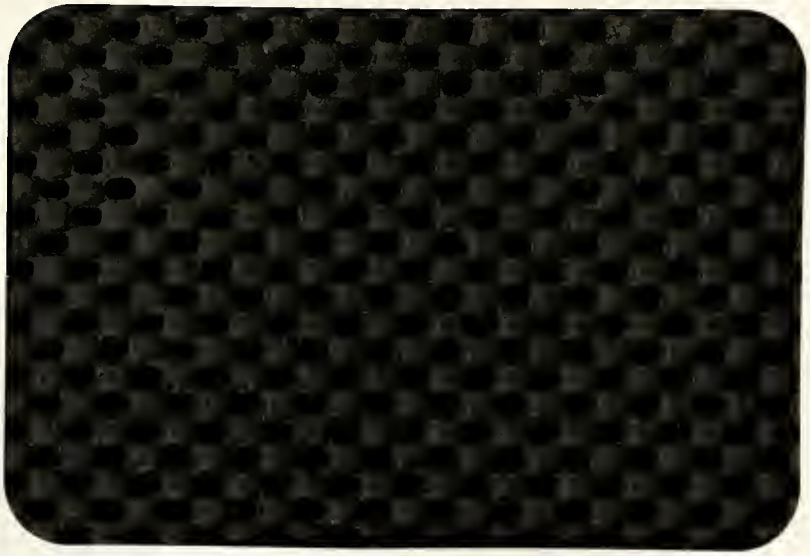

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On Intersection of Planar Jordan Curves

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ABSTRACT

Let $\gamma_1, \ldots, \gamma_n$ be n simple Jordan curves in the plane, and let $K_1, \ldots, K_n$ be their respective interior regions. It is shown that if each pair of curves γ_i, γ_j, $i \neq j$, intersect one another in at most two points, then the boundary of $K = \bigcup_{i=1}^{n} K_i$ contains at most $\max(2, 6n-12)$ intersection points of the curves γ_i. We also show that this bound is tight, and discuss some generalizations to surfaces of higher dimensions.

In this paper we will prove the property stated in the abstract. This property is a generalization of a similar one proved in [KS] for the special case in which the sets $K_1, \ldots, K_n$ are of the form $K_i = A_i + B$, where $A_1, \ldots, A_n$ are interior-disjoint closed convex sets, B is a closed convex set, and $A_i + B$ denotes the vector (Minkowski) sum of A_i and B. This special form of the theorem was needed for obtaining efficient algorithms for planning a purely translational motion of a convex polygonal object free to translate in a two-dimensional open space bounded by a collection of (convex) obstacles (see [KS] for more detail). We show here that this property is in fact purely topological, and that the special form of the sets K_i

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$$\partial D_i = \gamma_i - \left(\bigcup_{j \neq i} I_{ij}\right) \cup \left(\bigcup_{j \neq i} I_{ji}\right) .$$

For each i choose an arbitrary point $p_i \in int\, D_i$, and for each $j \neq i$, if $\gamma_j \cap \gamma_i \neq \emptyset$ choose an arbitrary point $q_{ij} \in I_{ji}$ which is not an endpoint. Then we can connect p_i to all such q_{ij} by simple arcs α_{ij} which are pairwise non-intersecting except at p_i, and contained in $int\, D_i$ except for their endpoints q_{ij}. Likewise we can connect q_{ij} to q_{ji} by a simple arc $\beta_{ij} = \beta_{ji}$ contained in the interior of the simple closed Jordan curve $I_{ij} \cup I_{ji}$ (this interior being also equal to $int\, \gamma_i \cap int\, \gamma_j$). See Fig. 1 for an illustration of these concepts.

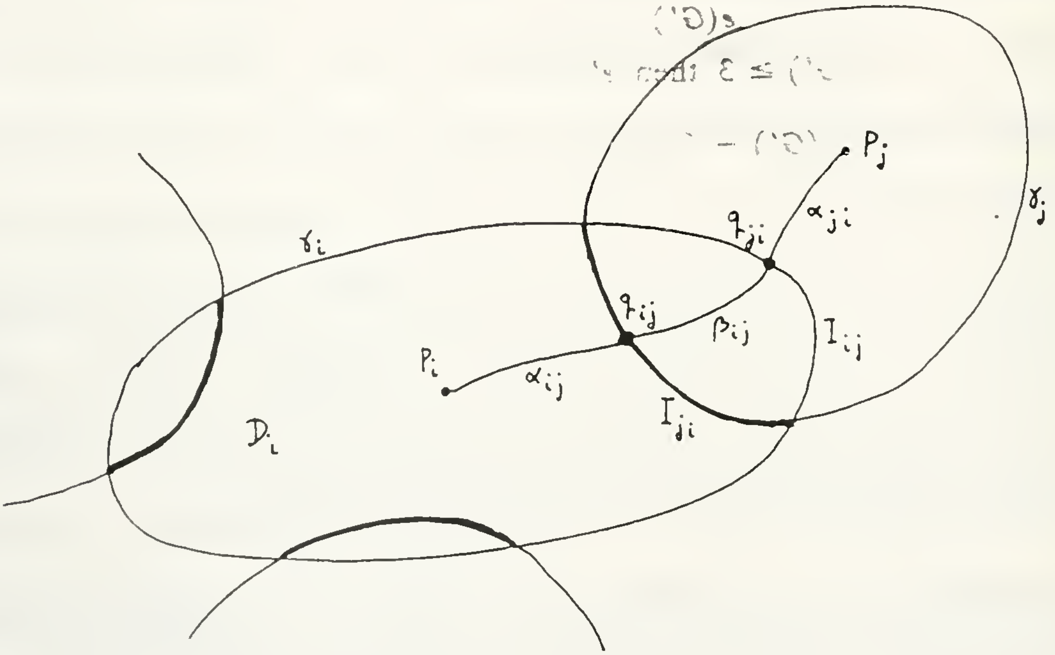

Fig. 1.

We obtain this way a graph $G(\Gamma)$ whose vertices are the points p_i such that for each $i \neq j$ for which $\gamma_i \cap \gamma_j \neq \emptyset$, $G(\Gamma)$ contains the edge $\alpha_{ij} \cup \beta_{ij} \cup \alpha_{ji}$ connecting p_i with p_j. The above discussion plainly implies that $G(\Gamma)$ is a planar graph.

Choose a connected component G' of $G(\Gamma)$ with $v(G')$ vertices, $e(G')$ edges and $f(G')$ (internal) faces. Since any two vertices of $G(\Gamma)$ are connected by at most one edge, each face of G' is bounded by at least three edges. Hence

$$2e(G') - e'(G') \geq 3f(G')$$

where $e'(G')$ is the number of edges of G' on the boundary of the external face of G'. By Euler's formula we have

$$v(G') - e(G') + f(G') = 1 .$$

Hence

$$e(G') \leq 3v(G') - 3 - e'(G') .$$

Note that if $v(G') \geq 3$ then $e'(G') \geq 3$, so that for such components we have $e(G') \leq 3v(G') - 6$. On the other hand, if $G(\Gamma)$ contains connected components G' with $v(G') < 3$ then, since we have assumed $n \geq 3$, $G(\Gamma)$ must contain at least two components G', each satisfying $e(G') \leq 3v(G') - 3$. In any case, summing the above inequalities over all connected components of $G(\Gamma)$, we obtain

$$e(G(\Gamma)) \leq 3v(G(\Gamma)) - 6 .$$

But $v(G(\Gamma)) = n$ and $e(G(\Gamma)) = \frac{1}{2} \# E(\Gamma)$. This implies the proposition.

Q.E.D.

Next assume Γ is admissible and the Theorem true for all admissible collections preceding Γ lexicographically.

Step 1: Assume $r_0(\Gamma) > 0$. Let p be a point lying on at least three of the curves in Γ. Without loss of generality we may assume that $p \in \bigcap_{i=1}^{k} \gamma_i$, and that $p \notin \gamma_j$ for $j > k$, for some $k \geq 3$.

Lemma 3: Given the above configuration, we may assume, after applying a homeomorphism of the plane onto itself,

1. p is the origin.

2. For each $i \leq k$ the intersection of γ_i with the unit disc $D = \{z \in C \mid |z| \leq 1\}$ is a diameter through p.

3. $\gamma_i \cap D = \varnothing$ for $i > k$.

(See Fig. 2.)

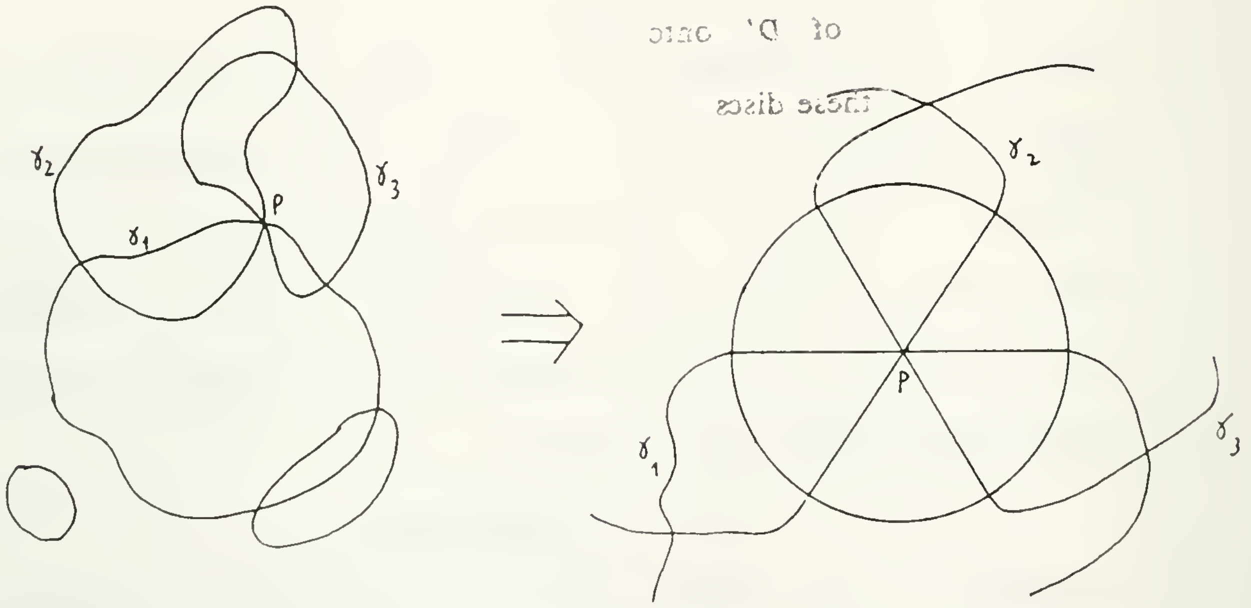

Fig. 2.

Proof: Choose a sufficiently small closed disc D^* around p so that $I(\Gamma) \cap D^* = \{p\}$, $D^* \cap K(\gamma_i) = \varnothing$ for $i > k$ and such that no γ_i is contained in D^*. For $i \leq k$ let δ_i be the path component of p in $\gamma_i \cap D^*$, and let δ_i^* be the closure of $\gamma_i - \delta_i$. Then $\delta_i \cap \delta_i^* = \delta_i \cap \partial D^*$ consists of two points p_i, q_i. Since by assumption every intersection point of the curves in Γ is a crossing one, it follows that for $i,j \leq k$, $i \neq j$ the pair $\{p_i, q_i\}$ separates the pair $\{p_j, q_j\}$ on ∂D^*. Hence we may assume that the points p_i, q_i are ordered along ∂D^* in counterclockwise order as $\{t_i\}_{i=1}^{2k}$, where $t_i = p_i$ and $t_{i+k} = q_i$ for $i = 1, \ldots, k$. Thus D^* is divided by the arcs δ_i into $2k$ sectors, all having p as apex. By Schoenfliess' Theorem we can map these sectors homeomorphically onto the $2k$ standard sectors $\frac{\pi(i-1)}{k} \leq arg\ z \leq \frac{\pi i}{k}$ of D in counterclockwise order so that these homeomorphisms glue together (along appropriate portions of the δ_i's) to a homeomorphism of D' onto D, which we can then extend to the complements of these discs to obtain a homeomorphism h of the whole plane onto itself. Finally, $p \notin \delta_i^*$, so that for a sufficiently small $\epsilon > 0$, $h(\delta_i^*) \cap \epsilon D = \emptyset$. Thus, replacing h by $h^* = \frac{1}{\epsilon}h$ we obtain the claim of the lemma. Q.E.D.

Using Lemma 3, it now follows that $D - \cup K(\gamma_i)$ is a (possibly empty) sector of D. Deforming each δ_i while keeping $\delta_i \cap \partial D$ fixed, we can keep the collection admissible with the same number of curves, not decrease $\# E(\Gamma)$, and reduce $r_0(\Gamma)$. The induction step in this case is therefore completed.

Step 2: Suppose $\Gamma = \{\gamma_i\}_{i=1}^{n}$ is admissible, $r_0(\Gamma) = 0$, and $r_1(\Gamma) > 0$. Choose a redundant curve γ_i in Γ, and replace Γ by $\Gamma^* = \Gamma - \{\gamma_i\}$. Clearly $r_0(\Gamma') = 0$ and $\# E(\Gamma') = \# E(\Gamma)$, while $r_1(\Gamma') < r_1(\Gamma)$, so that the induction step in this case is also completed.

Step 3: Suppose finally that $\Gamma = \{\gamma_i\}_{i=1}^{n}$ is admissible, $r_0(\Gamma) = 0$, $r_1(\Gamma) = 0$, and $r_3(\Gamma) > 0$. Without loss of generality we may assume that $K(\gamma_1) \cap K(\gamma_2) \cap K(\gamma_3) \neq \emptyset$, and that for some $k \geq 3$ we have $\gamma_1 \cap K(\gamma_i) = I_i \neq \emptyset$ for $i \leq k$, and $\gamma_1 \cap \gamma_i = \emptyset$ for $i > k$.

Lemma 4: Let $A = \{z \in C \mid \frac{1}{2} \leq |z| \leq 2\}$. After applying a homeomorphism of the plane onto itself we may assume

1. γ_1 is the unit circle.

2. For $i > k$, $K(\gamma_i) \cap A = \emptyset$.

3. For $i = 2, \ldots, k$ we have $K(\gamma_i) \cap A = \{z \in A \mid a_i \leq arg\ z \leq b_i\}$ for appropriate orientations a_i, b_i.

(See Fig. 3.)

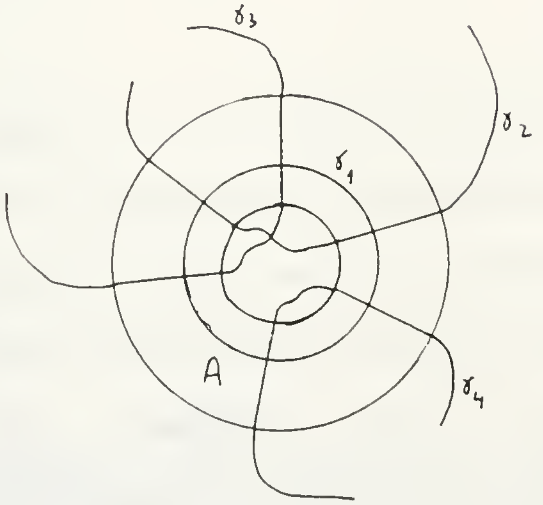

Fig. 3.

Proof: Using Schoenfliess' Theorem, we may assume that γ_1 is the unit circle $S^1 = \partial D$. Choose a sufficiently narrow closed annulus A^* around S^1 so that $A^* \cap I(\Gamma) \subset S^1$, $A^* \cap K(\gamma_i) = \emptyset$ for $i > k$, and such that A^* does not contain $K(\gamma_i)$ for $i \neq 1$. Since $r_0(\Gamma) = 0$ (i.e. no three curves in Γ have a common point) it follows that $I(\Gamma) \cap S^1$ consists of $l = 2k - 2$ distinct points $p_1, \ldots, p_l$. As in the proof of Lemma 3 we may now take δ_i to be the path component of that γ_j passing through p_i inside A^*, then apply a homeomorphism of the plane so as to make each δ_i a straight segment of the form $\delta_i = \{z \in A \mid arg\ z = c_i\}$ for appropriate orientations c_i, and finally shrink A^* and change scale, to obtain the claim in the lemma. Q.E.D.

Suppose now that Γ is as in Lemma 4, so that $\{I_i\}_{i=2}^{k}$ are now angular intervals on S^1.

Subcase 3.1: If for some $i = 2, \ldots, k$ the interval I_i is a subset of $\bigcup_{j=2,\, j \neq i}^{k} I_j$, replace γ_i by

$$\gamma_i^* = \gamma_i - \{z \in C \mid |z| \leq \frac{3}{2}\} \cup \{z \in C \mid |z| = \frac{3}{2}\} .$$

(See Fig. 4.)

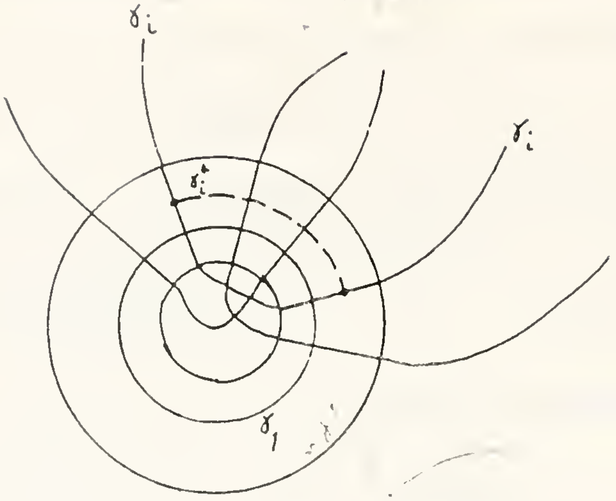

Fig. 4.

The resulting collection still has $r_0 = r_1 = 0$, but r_2 is smaller. Indeed $\gamma_i^* \cap \gamma_1 = \emptyset$ whereas $\gamma_i \cap \gamma_1 \neq \emptyset$, and no new intersections are added: if γ_i^* intersects some γ_j for $j \neq 1,i$ so did γ_i. Furthermore $\#E(\Gamma)$ has not changed, since the two eliminated intersections did not belong to $E(\Gamma)$. Thus the induction step can now be completed in this case.

Subcase 3.2: Suppose finally that Γ is as above, but that for each $i = 2, \ldots, k$ the interval I_i is not contained in $\bigcup_{j=2,\, j\neq i}^{k} I_j$. Since $r_1(\Gamma) = 0$ we may assume that $1 \in S^1 - \bigcup_{i \neq 1} K(\gamma_i)$. Let $I_i = \{z \in S^1 \mid a_i \leq arg\ z \leq b_i\}$. Since $r_0(\Gamma) = 0$ the a_i's and b_i's are all distinct. Also, since we now assume that no I_i is contained in $\bigcup_{j=2,\, j\neq i}^{k} I_j$, we may assume that $a_2 < a_3 < b_2 < b_3$, and also that $0 < a_2 < a_3 < \cdots < a_k < 2\pi$. But then we can replace γ_2 by

$$\gamma_2^* = \gamma_2 - \{z \mid |z| \leq \frac{3}{4}\} - \{z \mid |z| \leq \frac{3}{2}, arg\ z = b_2\}$$

$$\cup \{z \mid |z| = \frac{3}{4}, a_2 \le arg\ z \le \frac{a_2 + a_3}{2}\} \cup \{z \mid \frac{3}{4} \le |z| \le \frac{3}{2}, arg\ z = \frac{a_2 + a_3}{2}\}$$
$$\cup \{z \mid |z| = \frac{3}{2}, \frac{a_2 + a_3}{2} \le arg\ z \le b_2\} .$$

In other words, we pull γ_2 and γ_3 somewhat apart, so as to move their intersection point within $K(\gamma_1)$ to the exterior of that region (see Fig. 5).

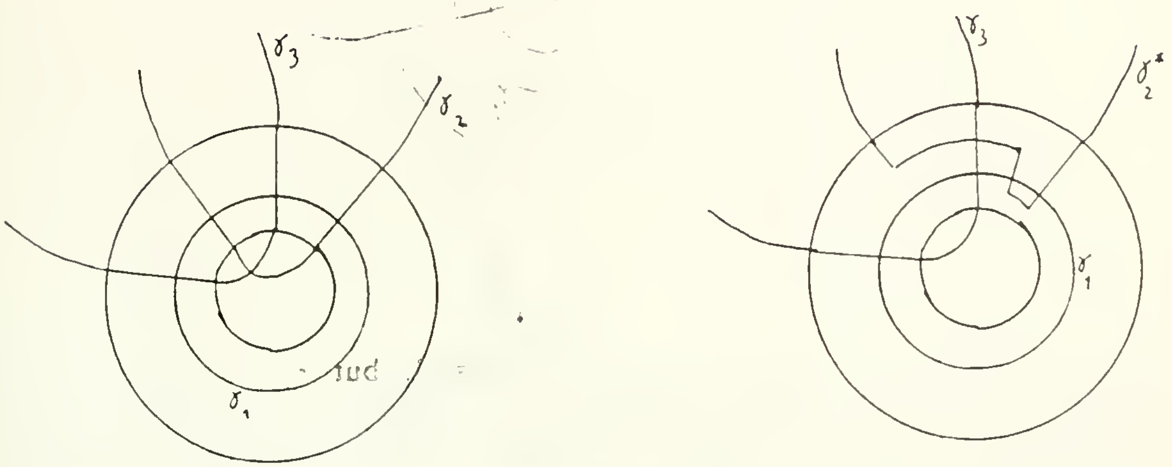

Fig. 5.

It is easily seen that this deformation of γ_2 keeps $r_0(\Gamma)$ and $r_1(\Gamma)$ both equal to zero, and does not increase $r_2(\Gamma)$. Furthermore, $r_3(\Gamma)$ has decreased, because now $K(\gamma_1) \cap K(\gamma_2) \cap K(\gamma_3) = \emptyset$, whereas $\#E(\Gamma)$ has not decreased. Thus the induction step is completed in this case too, thereby completing the proof of the first part of the theorem.

Next we show that the bound obtained is actually tight in the worst case. We will give such an example in which the curves in Γ are all circles (Γ is always (weakly) admissible in this case). We draw $n \ge 3$ circles in the plane as follows. The first three circles are drawn so that each pair of them intersect, and such that their union is homeomorphic to an annulus, i.e. has a hole in the center. Then, inductively, we draw each additional circle inside a hole bounded by three arcs of previously drawn circles so that it intersects each of those arcs at two points, thereby leaving three smaller holes out of the original hole (see Fig. 6).

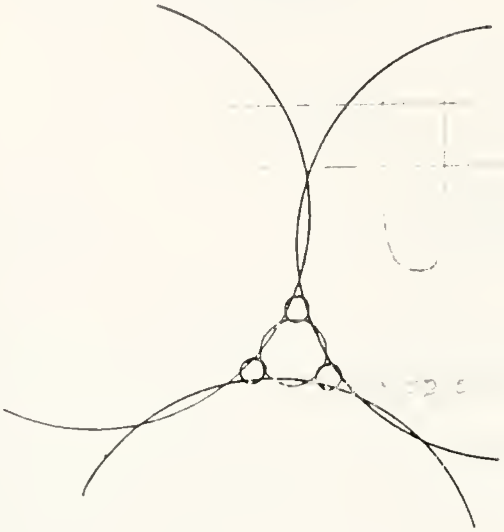

Fig. 6.

Thus the first three circles contribute six points to $E(\Gamma)$ and each additional circle contributes six more points, yielding altogether $6n - 12$ points in $E(\Gamma)$. This completes the proof of the theorem. Q.E.D.

Remarks: (1) If we relax the requirement that each pair of the curves in Γ intersect in at most two points, the size of $E(\Gamma)$ can increase significantly. For example, if each pair of curves in Γ is allowed to intersect as much as a maximum of four times, $\# E(\Gamma)$ can become $\Omega(n^2)$, as shown in Fig. 7.

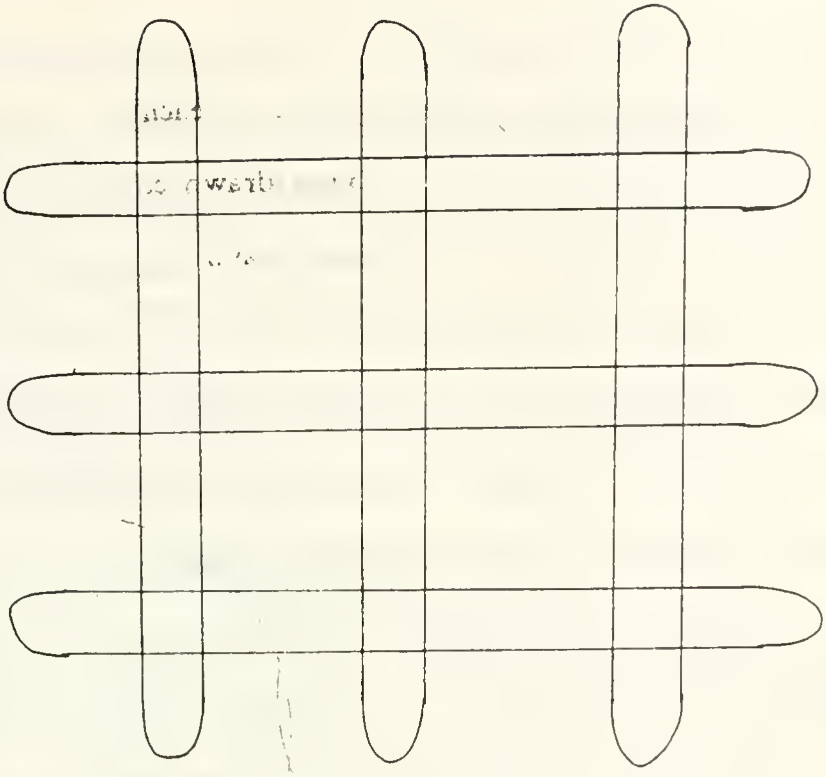

Fig. 7.

(2) Theorem 1 can be generalized to the case of Jordan curves on the unit sphere S^2, where each curve γ_i partitions S^2 into an "interior" region and an "exterior" one. Indeed, if the union K of the closures $K(\gamma_i)$ of the interiors of the curves γ_i is the whole of S^2 then there is nothing to prove. Otherwise choose a point p in $S^2 - K$ and map $S^2 - \{p\}$ onto the plane, thereby reducing the situation to that assumed in Theorem 1.

(3) As already noted, if each curve γ_i is a circle then clearly $\Gamma = \{\gamma_i\}_{i=1}^n$ is admissible. In this special case however the property established in Theorem 1 is already known; it follows for example from the properties of Voronoi diagrams for a set of intersecting discs in the plane (cf. [Sh]).

(4) If each of the curves γ_i in an admissible collection Γ has a relatively simple shape, we can calculate the union $K(\Gamma)$ in an efficient manner, generalizing the algorithm given in [KS]. Roughly speaking, this calculation

proceeds by dividing the collection $\Gamma = \{\gamma_i\}_{i=1}^n$ into two subcollections Γ_1, Γ_2, each containing about half the number of curves. Then we calculate recursively the two sets $K(\Gamma_1)$, $K(\Gamma_2)$, and finally calculate their union, using a sweeping line technique. The computational complexity of this algorithm depends of course on the complexity of each of the curves in Γ, but if one assumes e.g. that each $K(\gamma_i)$ is convex and has simple enough shape so that it takes a constant amount of time to find its leftmost and rightmost points, to calculate the intersection points of any pair of curves in Γ, and to determine whether a given point lies inside or outside a region $K(\gamma_i)$, then the above algorithm can be implemented to run in time $O(n \log^2 n)$. See [KS] for more details. For example, if Γ is a collection of circles, then this algorithm yields an alternative method of calculating $K(\Gamma)$, which has the same computational complexity as the Voronoi diagram-based technique given in [Sh].

Generalization to higher dimensional surfaces

Let $\Sigma = \{\sigma_i\}_{i=1}^n$ be a collection of two dimensional surfaces in E^3 which satisfy the following properties:

(i) Each σ_i is homeomorphic to the unit sphere S^2.

(ii) For each $i \neq j$, $\sigma_i \cap \sigma_j$ is either homeomorphic to S^1 or is empty.

(iii) For each triple i,j,k of distinct indices $\sigma_i \cap \sigma_j \cap \sigma_k$ is either homeomorphic to S^0 or is empty.

For example, if each of the σ_i is a sphere then conditions (i)-(iii) hold, assuming non-degeneracy of the configuration of these spheres (that is, assuming that no pair of spheres is tangent to one another and that no sphere is tangent to the intersection curve of two other spheres; nevertheless, as in the 2-D case, Theorem 5 below will also hold is such a degeneracy occurs).

Let $K(\sigma_i)$ be the closure of the interior region bounded by σ_i, and let $K(\Sigma) = \bigcup_{i=1}^{n} K(\sigma_i)$. Denote by $I(\Sigma)$ the set of all points of triple intersection of the surfaces in Σ, and let $E(\Sigma) = I(\Sigma) \cap \partial K(\Sigma)$. We have trivially $\# I(\Gamma) \leq \frac{n(n-1)(n-2)}{3}$ and this bound can be easily achieved for an appropriate collection of n spheres. However we have

Theorem 5: If $\Sigma = \{\sigma_i\}_{i=1}^{n}$, $n \geq 4$, is a collection of two dimensional surfaces in 3-space satisfying conditions (i)-(iii) listed above, then $\# E(\Sigma) \leq 2n(n-3)$.

Proof: Fix one of the surfaces σ_i and consider the collection of curves $\gamma_j = \sigma_i \cap \sigma_j$, $j \neq i$, drawn on it. Some of these curves can be empty, but each nonempty γ_j is a closed Jordan curve on σ_i, and for each $j \neq k \neq i$ we have $\#(\gamma_i \cap \gamma_j) \leq 2$. Hence we can define $K(\gamma_j) = \sigma_i \cap K(\sigma_j)$, for $j \neq i$, and apply Theorem 1 (or rather its generalization as in Remark (2) above) to the collection $\Gamma = \{\gamma_j\}_{j \neq i}$, to obtain $\#(E(\Sigma) \cap \sigma_i) \leq 6(n-1) - 12 = 6(n-3)$. Repeating this argument for each of the surfaces σ_i, and observing that each point in $E(\Sigma)$ will be counted that way three times, we conclude that $\# E(\Sigma) \leq \frac{n}{3} \cdot 6(n-3) = 2n(n-3)$.

Q.E.D.

Corollary: Given n distinct balls $B_1, \ldots, B_n$ in 3-space, no pair of which are tangent to one another, the number of triple intersections of these balls which lie on the boundary of their union is at most $2n(n - 3)$.

(This property certainly holds if the configuration of the spheres is non-degenerate, and can be easily inferred by continuity arguments for degenerate configurations as well.)

Remark: We do not know whether this bound is tight. However, even when the surfaces in Σ are all spheres, $E(\Sigma)$ can contain $\Omega(n^2)$ points, so that in the worst case the above bound is tight up to a multiplicative constant. To see such an example, let $n = 2k$; let $\sigma_1, \ldots, \sigma_k$ be k spheres whose centers are the points $(\cos\frac{2\pi i}{k}, \sin\frac{2\pi i}{k}, 0)$, $i = 1, \ldots, k$, and whose radii are all equal to some r_0 lying strictly between $\sin\frac{\pi}{k}$ and 1, so that the boundary of the union $K_0 = \bigcup_{i=1}^{k} K(\sigma_i)$ is homeomorphic to a torus. Note that the intersection curves of these first k spheres, which lie on the boundary of K_0, are k circles $\gamma_1, \ldots, \gamma_k$ of equal radii, such that the planes containing them all pass through the z-axis, and such that all these circles can be obtained by a revolution of one of them about the z-axis. The remaining k spheres are arranged so that their centers all lie on the z-axis, and such that each of them intersects each of the curves γ_i at a pair of points. By choosing the centers and the radii of these k additional spheres in an appropriate manner, we can ensure that each of these intersection points belongs to $E(\Sigma)$, so that $\#E(\Sigma) = 2k^2 = \frac{n^2}{2}$.

Remark: Theorem 5 can be generalized to surfaces of arbitrary dimension, in a straightforward way. In particular, this generalization will imply that for any collection of n $(d-1)$-spheres in E^d, the total number of points of intersection of d of these spheres which lie on the boundary of their union is $O(n^{d-1})$.

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